An observation on the Turán-Nazarov inequality

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Abstract

The main observation of this note is that the Lebesgue measure μ in the Turán-Nazarov inequality for exponential polynomials can be replaced with a certain geometric invariant $\omega \geq \mu$, which can be effectively estimated in terms of the metric entropy of a set, and may be nonzero for discrete and even finite sets. While the frequencies (the imaginary parts of the exponents) do not enter in the original Turán-Nazarov inequality, they necessarily enter the definition of ω.

1 Introduction

The classical Turán inequality bounds the maximum of the absolute value of an exponential polynomial $p(t)$ on an interval B through the maximum of its absolute value on any subset Ω of positive measure. Turán [8] assumed Ω to be a subinterval of B, and Nazarov [4] generalized it to any subset Ω of positive measure. More precisely, we have:

Theorem 1.1 ([4]). *Let $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$ be an exponential polynomial, $c_k, \lambda_k \in \mathbb{C}$. Let $B \subset \mathbb{R}$ be an interval, and let $\Omega \subset B$ be a measurable set. Then*

$$\sup_B |p| \leq e^{\mu_1(B)\cdot \max |\operatorname{Re}\lambda_k|} \cdot \left(\frac{c\mu_1(B)}{\mu_1(\Omega)}\right)^m \cdot \sup_\Omega |p|$$

where μ_1 is the Lebesgue measure on $\mathbb{R}$ and $c > 0$ is an absolute constant.

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In this note, we generalize and strengthen the Turán-Nazarov inequality (and its multi-dimensional analogue stated below) by replacing the Lebesgue measure of Ω with a simple geometric invariant $\omega_D(\Omega)$, the metric span of $\Omega \subset \mathbb{R}^n$ with respect to a "diagram" D comprising the degree of p and its maximal frequency λ. The metric span always bounds the Lebesgue measure from above, and it is strictly positive for sufficiently dense discrete (in particular, finite) sets Ω. It can be effectively estimated in terms of the metric entropy of Ω. See [10] and Section 2.1 below for some basic properties of $\omega_D(\Omega)$. A somewhat simpler version of the metric span of Ω depending only on the dimension and the degree, and not on the continuous parameters, was originally introduced in [10]. It replaces the Lebesgue measure of Ω in the classical Remez inequality for algebraic polynomials ([6, 2]). In the one-dimensional case for a given exponential polynomial $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$, $c_k, \lambda_k \in \mathbb{C}$, and for a given interval $B \subset \mathbb{R}$ the diagram $D = D(p, B)$ comprises the degree m, the length $\mu_1(B)$ and the maximal frequency $\lambda = \max\limits_{k=0,\dots,m} |\operatorname{Im} \lambda_k|$. Define the constant M_D (which we call a "frequency bound" for p) as $M_D = \lfloor \frac{d}{2} \rfloor + 1$, where $d = C(m)\mu_1(B)\lambda$. Here $C(m)$ is defined as $C(m) = n(2n+1)^{2n}2^{2n^2}$, for $n = \frac{(m+1)(m+2)}{2} + 1$. For any bounded subset $\Omega \subset \mathbb{R}$ and for $\varepsilon > 0$ let $M(\varepsilon, \Omega)$ be the minimal number of ε-intervals covering Ω (which are translations of $[0, \varepsilon]$). Now the metric span ω_D is defined as follows:

Definition 1.1. The metric span $\omega_D(\Omega)$ of $\Omega \subset \mathbb{R}$ is given by

$$\omega_D(\Omega) = \sup_{\varepsilon > 0} \varepsilon [M(\varepsilon, \Omega) - M_D].$$

Now we can state our main result in the one-dimensional case:

Theorem 1.2. *Let $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$ be an exponential polynomial, $c_k, \lambda_k \in \mathbb{C}$. Let $B \subset \mathbb{R}$ be an interval, and let $\Omega \subset B$ be any set. Then*

$$\sup_B |p| \leq e^{\mu_1(B) \cdot \max |\operatorname{Re} \lambda_k|} \cdot \left(\frac{c\mu_1(B)}{\omega_D(\Omega)} \right)^m \cdot \sup_\Omega |p|$$

where μ_1 is the Lebesgue measure on $\mathbb{R}$ and $c > 0$ is an absolute constant.

Clearly, for any *measurable* Ω we always have $\omega_D(\Omega) \geq \mu_1(\Omega)$. Indeed, for any $\varepsilon > 0$ we have $M(\varepsilon, \Omega) \geq \mu_1(\Omega)/\varepsilon$. Now substitute into Definition 1.1 and let ε tend to zero. Thus, Theorem 1.2 provides a true generalization and strengthening of the Turán-Nazarov inequality given in Theorem 1.1.

Moreover, the result of Theorem 1.2 further develops a remarkable feature of the original Turán-Nazarov inequality: The bound does not depend on the "frequencies", i.e. on the imaginary parts of λ_k in p. When we allow into consideration *discrete* (in particular, *finite*) sets Ω, this feature cannot be preserved: Already for a trigonometric polynomial $p(t) = \sin(\lambda t)$, the set Ω of its zeroes (on which the Turán-Nazarov inequality certainly fails) consists of all the points $x_j = \frac{j\pi}{\lambda}, \ j \in \mathbb{N}$, and the number of such points in any interval B is of order $\frac{\mu_1(B)\lambda}{\pi}$. So when we replace the Lebesgue measure with the metric span, we have to take into account the imaginary parts of the exponents λ_k. This is exactly what is done in Definition 1.1 and in Theorem 1.2 above. Thus, our result separates the roles of the real and imaginary parts of the exponents: The first enters in the main bound, as in the original Turán-Nazarov inequality, while the second enters in the definition of the span $\omega_D(\Omega)$. As the density of Ω grows, the influence of the frequencies decreases: See Section 2.1 below.

There is a version of Turán-Nazarov inequality for quasipolynomials in one or several variables due to A. Brudnyi [1, Theorem 1.7]. While less accurate than the original one (in particular, the role of real and complex parts of the exponents is not separated) this result gives an important information for a wider class of quasipolynomials. In Section 3 we provide a strengthening of Brudnyi's result in the same lines as above: We replace the Lebesgue measure with an appropriate "metric span" which always bounds the Lebesgue measure from above and is strictly positive for sufficiently dense discrete (in particular, finite) sets.

2 One-dimensional case

In this section we prove Theorem 1.2 and provide some of its consequences.

Proof of Theorem 1.2. Let $p(t) = \sum_{k=0}^{m} c_k e^{\lambda_k t}$ be an exponential polynomial, $c_k, \lambda_k \in \mathbb{C}$. Let us write $c_k = \gamma_k e^{i\phi_k}, \ \lambda_k = a_k + ib_k, \ k = 0, 1, \ldots, m$.

Lemma 2.1.

$$|p(t)|^2 = 2 \sum_{0 \le k \le l \le m} \gamma_k \gamma_l e^{(a_k + a_l)t} \cos(\phi_k - \phi_l + (b_k - b_l)t)$$

is an exponential trigonometric polynomial of degree $\frac{(m+1)(m+2)}{2}$ *with real coefficients.*

Proof. We have

$$p(t) = \sum_{k=0}^{m} \gamma_k e^{i\phi_k} e^{(a_k+ib_k)t} = \sum_{k=0}^{m} \gamma_k e^{a_k t + i(\phi_k + b_k t)},\ \bar{p}(t) = \sum_{k=0}^{m} \gamma_k e^{a_k t - i(\phi_k + b_k t)}$$

Therefore

$$|p(t)|^2 = p(t)\bar{p}(t) = \sum_{k,l=0}^{m} \gamma_k \gamma_l e^{(a_k+a_l)t + i(\phi_k - \phi_l + (b_k - b_l)t)}$$

Adding the expressions in this sum for the indices (k, l) and (l, k) we get

$$|p(t)|^2 = 2\sum_{k\le l} \gamma_k \gamma_l e^{(a_k+a_l)t} \cos(\phi_k - \phi_l + (b_k - b_l)t)$$

This completes the proof. ■

The following lemma provides us with a bound on the number of real solutions of the equation $|p(t)|^2 = \eta$. It is a direct consequence of Theorem 3.3 and Lemma 3.4, see Section 3.1 below.

Lemma 2.2. *For $p(t)$ as above and for each positive $\eta > 0$, the number of non-degenerate solutions of the equation $|p(t)|^2 = \eta$ in the interval $B \subset \mathbb{R}$ does not exceed*

$$d = C(m)\mu_1(B)\lambda$$

where $\lambda = \max|\operatorname{Im}\lambda_k|$, and $C(m) = n(2n+1)^{2n}2^{2n^2}$, for $n = \frac{(m+1)(m+2)}{2} + 1$.

Let $B \subset \mathbb{R}$ be an interval. We consider the sublevel set $V_\rho = \{t \in B : |p(t)| \le \rho\}$ of an exponential polynomial $p(t) = \sum_{k=0}^{m} c_k e^{\lambda_k t}$, $c_k, \lambda_k \in \mathbb{C}$. By Lemma 2.2 the boundary of V_ρ given by $\{|p(t)|^2 = \rho^2\}$ consists of at most $d = C(m)\mu_1(B)\max|\operatorname{Im}\lambda_k|$ points (including the endpoints). Therefore, the set V_ρ consists of at most $M_D = \lfloor \frac{d}{2} \rfloor + 1$ subintervals Δ_i (i.e. connected components of V_ρ), with M_D defined as in Theorem 1.2. Let us cover each of these subinterval Δ_i by the adjacent ε-intervals Q_ε starting with the left endpoint. Since all the adjacent ε-intervals, except possibly one, are inside Δ_i, their number doesn't exceed $|\Delta_i|/\varepsilon + 1$. Thus, we have

$$M(\varepsilon, V_\rho) \le (\lfloor \frac{d}{2} \rfloor + 1) + \mu_1(V_\rho)/\varepsilon = M_D + \mu_1(V_\rho)/\varepsilon$$

using the notations of Theorem 1.2. Now let a set $\Omega \subset B$ be given.

Lemma 2.3. *If $\Omega \subset V_\rho$ for a certain $\rho \geq 0$ then $\mu_1(V_\rho) \geq \omega_D(\Omega)$.*

Proof. If $\Omega \subset V_\rho$ then for each $\varepsilon > 0$ we have $M(\varepsilon, \Omega) \leq M(\varepsilon, V_\rho) \leq M_D + \mu_1(V_\rho)/\varepsilon$, or $\mu_1(V_\rho) \geq \varepsilon(M(\varepsilon, \Omega) - M_D)$. Taking supremum with respect to $\varepsilon > 0$ and using Definition 1.1 we conclude that $\mu_1(V_\rho) \geq \omega_D(\Omega)$. ■

Let us now put $\hat{\rho} = \sup_\Omega |p|$. Then by the definition we have $\Omega \subset V_{\hat{\rho}}$. Applying Lemma 2.3 we get $\mu_1(V_{\hat{\rho}}) \geq \omega_D(\Omega)$. Finally, we apply the original Turán-Nazarov inequality (Theorem 1.1) to the subset $V_{\hat{\rho}} \subset B$ on which $|p|$ by definition does not exceed $\hat{\rho}$. This completes the proof of Theorem 1.2. ■

We expect that the expression for $C(m)$ in Lemma 2.2 provided by the general result of Khovanskii can be strongly improved in our specific case. Let us recall the following result of Nazarov [4, Lemma 4.2], which gives a much more realistic bound on the local distribution of zeroes of an exponential polynomial:

Lemma 2.4. *Let $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$ be an exponential polynomial, $c_k, \lambda_k \in \mathbb{C}$. Then the number of zeroes of $p(z)$ inside each disk of radius $r > 0$ does not exceed $4m + 7\hat{\lambda}r$, where $\hat{\lambda} = \max|\lambda_k|$.*

The reason we use the Khovanskii bound in Theorem 1.2 is that it involves only the imaginary parts of the exponents λ_k. In contrast, the bound of Lemma 2.4 is in terms of $\hat{\lambda} = \max|\lambda_k|$ (as opposed to $\max|\operatorname{Im}\lambda_k|$). In order to apply Lemma 2.4 we notice that

$$|p(t)|^2 = p(t)\bar{p}(t) = \sum_{k,l=0}^m c_k \bar{c}_l e^{(\lambda_k + \bar{\lambda}_l)t}$$

is an exponential polynomial of degree at most m^2 with the maximal absolute value of the exponents not exceeding $2\hat{\lambda}$. Adding a constant adds at most one to the degree. We conclude that the number of real solutions of $|p(t)|^2 = \eta$ inside the interval B does not exceed $d_1 = 4m^2 + 14\hat{\lambda}\mu_1(B)$. Now we define ω'_D putting $M'_D = \lfloor \frac{d_1}{2} \rfloor + 1$ in Definition 1.1. Repeating word by word the proof of Theorem 1.2 above we obtain:

Theorem 2.5. *For $p(t)$ as above*

$$\sup_B |p| \leq e^{\mu_1(B) \cdot \max|\operatorname{Re}\lambda_k|} \cdot \left(\frac{c\mu_1(B)}{\omega'_D(\Omega)}\right)^m \cdot \sup_\Omega |p|.$$

For the case of a real exponential polynomial $p(t) = \sum_{k=0}^{m} c_k e^{\lambda_k t}$, $c_k, \lambda_k \in \mathbb{R}$, we get an especially simple and sharp result. Notice that the number of zeroes of a real exponential polynomial is always bounded by its degree m (indeed, the "monomials" $e^{\lambda_k t}$ form a Chebyshev system on each real interval). Applying this fact in the same way as above we get

Theorem 2.6. *For $p(t)$ a real exponential polynomial of degree m*

$$\sup_B |p| \leq e^{\mu_1(B)\cdot \max|\lambda_k|} \cdot \left(\frac{c\mu_1(B)}{\omega''_D(\Omega)}\right)^m \cdot \sup_\Omega |p|$$

where $\omega''_D(\Omega) = \sup_{\varepsilon>0} \varepsilon[M(\varepsilon,\Omega) - m]$.

Notice that in this case the metric span $\omega''_D(\Omega)$ depends only on the degree m of p and the result is sharp: For any Ω consisting of at least $m+1$ points there is an inequality of the required form, while for each m points there is a real exponential polynomial $p(t)$ of degree m vanishing at exactly these points.

2.1 Some examples

In this section we give just a couple of examples illustrating the scope and possible applications of Theorem 1.2.

2.1.1 Subsets Ω dense "in resolution ε"

Here we show that the role of the frequency bound in the results above decreases as the discrete subset $\Omega \subset B$ becomes denser. For $\Omega \subset B$ and for $\varepsilon > 0$ we define the "measure $\mu_1(\varepsilon,\Omega)$ of Ω in resolution ε" as the minimal possible measure of the coverings of Ω with ε-intervals.

Proposition 2.7. *For each diagram D and for any $\varepsilon > 0$ the metric span $\omega_D(\Omega)$ satisfies*

$$\omega_D(\Omega) \geq \mu_1(\varepsilon,\Omega)\left(1 - \frac{\varepsilon M_D}{\mu_1(\varepsilon,\Omega)}\right)$$

Proof. By the definition $\omega_D(\Omega) \geq \varepsilon[M(\varepsilon,\Omega) - M_D]$. Clearly, $M(\varepsilon,\Omega) \geq \frac{1}{\varepsilon}\mu_1(\varepsilon,\Omega)$. Hence $\omega_D(\Omega) \geq \mu_1(\varepsilon,\Omega) - \varepsilon M_D$. ■

So if in a small resolution ε, the measure $\mu := \mu_1(\varepsilon,\Omega) > 0$ then we restore the original Turán-Nazarov inequality for Ω, with a correction factor $1 - \frac{\varepsilon M_D}{\mu}$, with M_D being the frequency bound.

2.1.2 Combining the discrete and positive measure cases

Let a diagram D be fixed, and let $\Omega = \Omega_1 \cup \Omega_2 \subset B$, with Ω_1 a set of a positive measure μ, and Ω_2 a discrete set. We assume that the sets Ω_1 and Ω_2 are $2\frac{\mu_1(B)}{M_D}$-separated, where M_D is the frequency bound for D.

Proposition 2.8. $\omega_D(\Omega) \geq \mu + \omega_D(\Omega_2)$

Proof. By the definition $\omega_D(\Omega) = \sup_\varepsilon \varepsilon[M(\varepsilon, \Omega) - M_D]$, and this supremum is achieved for $\varepsilon \leq \frac{\mu_1(B)}{M_D}$. Indeed, otherwise $M(\varepsilon, \Omega) - M_D$ would be negative. Hence by the separation assumption we have $M(\varepsilon, \Omega) = M(\varepsilon, \Omega_1) + M(\varepsilon, \Omega_2)$ and therefore $\omega_D(\Omega) = \sup_\varepsilon \varepsilon(M(\varepsilon, \Omega_1) + M(\varepsilon, \Omega_2) - M_D) \geq \mu_1(\Omega_1) + \omega_D(\Omega_2)$. ■

So in situations as above Theorem 1.2 improves the original Turán-Nazarov inequality, and the frequency bound applies only to the discrete part of Ω.

2.1.3 Interpolation with exponential polynomials

This is a classical topic starting at least with [5] and actively studied today in connection with numerous applications. Theorems 1.2, 2.5, 2.6 connect the Turán-Nazarov inequality on $\Omega \subset B$ with estimates for the robustness of the interpolation from Ω to B. In particular, they provide robustness estimates in solving the "generalized Prony system" for non-uniform samples. See [7] for some initial results in this direction.

3 Multi-dimensional case

In this section we consider the version of Turán-Nazarov inequality for quasipolynomials in one or several variables due to A. Brudnyi [1, Theorem 1.7]. We provide a strengthening of this result in the same lines as above: The Lebesgue measure is replaced with an appropriate "metric span". First, let us recall some definitions.

Definition 3.1. Let $f_1, \dots, f_k \in (\mathbb{C}^n)^*$ be a pairwise different set of complex linear functionals f_j which we identify with the scalar products $f_j \cdot z,\ z = (z_1, \dots, z_n) \in \mathbb{C}^n$. We shall write

$$f_j = a_j + ib_j$$

A quasipolynomial is a finite sum

$$p(z) = \sum_{j=1}^{k} p_j(z) e^{f_j \cdot z}$$

where $p_j \in \mathbb{C}[z_1, \dots, z_n]$ are polynomials in z of degrees d_j. The degree of p is $m = \deg p = \sum_{j=1}^{k}(d_j + 1)$. Following A.Brudnyi [1], we introduce the exponential type of p

$$t(p) = \max_{1 \le j \le k} \max_{z \in B_c(0,1)} |f_j \cdot z|$$

where $B_c(0,1)$ is the complex Euclidean ball of radius 1 centered at 0.

Below we consider $p(x)$ for the real variables $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

Theorem 3.1 ([1])**.** *Let p be a quasipolynomial with parameters n, m, k defined on $\mathbb{C}^n$. Let $B \subset \mathbb{R}^n$ be a convex body, and let $\Omega \subset B$ be a measurable set. Then*

$$\sup_B |p| \le \left(\frac{cn\mu_n(B)}{\mu_n(\Omega)} \right)^{\ell} \cdot \sup_\Omega |p|$$

where $\ell = (c(m,k) + (m-1)\log(c_1 \max\{1, t(p)\}) + c_2 t(p) \operatorname{diam}(B))$, and c, c_1, c_2 are absolute positive constants, and $c(k,m)$ is a positive number depending only on m and k.

Generalizing this result of Brudnyi, we follow the arguments described in Sections 1 and 2 above, and [10].

3.1 Covering number of sublevel sets

For a relatively compact $A \subset \mathbb{R}^n$, the covering number $M(\varepsilon, A)$ is defined now as the minimal number of ε-cubes Q_ε covering A (which are translations of the standard ε-cubes $Q_\varepsilon^n := [0, \varepsilon]^n$).

Lemma 3.2.

$$\begin{aligned} q(x) &:= |p(x)|^2 \\ &= \sum_{0 \le i \le j \le k} e^{\langle a_i + a_j, x \rangle} \left[P_{i,j}(x) \sin\langle b_i - b_j, x \rangle + Q_{i,j}(x) \cos\langle b_i - b_j, x \rangle \right] \end{aligned}$$

is a real exponential trigonometric quasipolynomial with $P_{i,j}, Q_{i,j}$ real polynomials in x of degree $d_i + d_j$, and at most $\kappa := k(k+1)/2$ exponents, sinus and cosinus elements.

Proof. By repeating word by word the proof of Lemma 2.1 above, the proof is completed. ∎

Clearly, all the partial derivatives $\frac{\partial q(x)}{\partial x_j}$ have exactly the same form. The following bound due to Khovanskii gives an estimate of the number of solutions of a system of real exponential trigonometric quasipolynomials. More precisely, we have

Theorem 3.3 (Khovanskii bound [3], Section 1.4)**.** *Let $P_1 = \cdots = P_n = 0$ be a system of n equations with n real unknowns $x = x_1, \ldots, x_n$, where P_i is polynomial of degree m_i in $n + k + 2p$ real variables x, $y_1, \ldots, y_k$, $u_1, \ldots, u_p$, $v_1, \ldots, v_p$, where $y_i = \exp\langle a_j, x\rangle$, $j = 1, \ldots, k$ and $u_q = \sin\langle b_q, x\rangle$, $v_q = \cos\langle b_q, x\rangle$, $q = 1, \ldots, p$. Then the number of non-degenerate solutions of this system in the region bounded by the inequalities $|\langle b_q, x\rangle| < \pi/2$, $q = 1, \ldots, p$, is finite and less than*

$$m_1 \cdots m_n \left(\sum m_i + p + 1\right)^{p+k} 2^{p+(p+k)(p+k-1)/2}$$

Let us denote the vectors $b_i - b_j \in \mathbb{R}^n$ by $b_{i,j}$ and let $\lambda := \max \|b_{i,j}\|$ be the maximal frequency in q. The next lemma is a simple consequence of Khovanskii bound:

Lemma 3.4. *Let V be a parallel translation of the coordinate subspace in $\mathbb{R}^n$ generated by $x_{j_1}, \ldots, x_{j_s}$. Then the number of non-degenerate real solutions in $V \cap Q^n_\rho$ of the system*

$$\frac{\partial q(x)}{\partial x_{j_1}} = \cdots = \frac{\partial q(x)}{\partial x_{j_s}} = 0$$

is at most $\hat{C}_s \lambda^s$, where

$$\hat{C}_s = (\frac{2}{\pi}\sqrt{s}\rho)^s \prod_{r=1}^{s}(d_{j_r} + d_{i_r}) \left(\sum_{r=1}^{s} d_{j_r} + d_{i_r} + 2\kappa + 1\right)^{2\kappa} 2^{\kappa+(2\kappa)(2\kappa-1)/2}.$$

Proof. The following geometric construction is required by the Khovanskii bound: Let $Q_{i,j} = \{x \in \mathbb{R}^n, |\langle b_{i,j}, x\rangle| \le \frac{\pi}{2}\}$ and let $Q = \bigcap_{0 \le i \le j \le k} Q_{i,j}$. For any $B \subset \mathbb{R}^n$ we define $M(B)$ as the minimal number of translations of Q covering B. For an affine subspace V of $\mathbb{R}^n$ we define $M(B \cap V)$ as the minimal number of translations of $Q \cap V$ covering $B \cap V$. Notice that for

$B = Q_r^n$, a cube of size r, we have $M(Q_r^n) \leq (\frac{2}{\pi}\sqrt{n}r\lambda)^n$. Indeed, Q always contains a ball of radius $\frac{\pi}{2\lambda}$. Now, applying the Khovanskii bound 3.3 on the system

$$\frac{\partial q(x)}{\partial x_{j_1}} = \cdots = \frac{\partial q(x)}{\partial x_{j_s}} = 0$$

we get that the number of non-degenerate real solutions in $V \cap Q_\rho^n$ is at most

$$(\frac{2}{\pi}\sqrt{s}\rho\lambda)^s \prod_{r=1}^{s}(d_{j_r} + d_{i_r}) \left(\sum_{r=1}^{s} d_{j_r} + d_{i_r} + 2\kappa + 1\right)^{2\kappa} 2^{\kappa+(2\kappa)(2\kappa-1)/2}$$

■

Let a quasipolynomial p be as above. A sublevel set $A = A_\rho$ of p is defined as $A = \{x \in \mathbb{R}^n : |p(x)| \leq \rho\}$. The following lemma extends to the case of sublevel sets of exponential polynomials the result of Vitushkin [9] for semi-algebraic sets. It can be proved using a general result of Vitushkin in [9] through the use of "multi-dimensional variations". However, in our specific case the proof below is much shorter and it produces explicit ("in one step") constants.

Lemma 3.5. *For any $1 \geq \varepsilon > 0$ we have*

$$M(\varepsilon, A \cap Q_1^n) \leq C_0 + C_1\left(\frac{1}{\varepsilon}\right) + \cdots + C_{n-1}\left(\frac{1}{\varepsilon}\right)^{n-1} + \mu_n(A)\left(\frac{1}{\varepsilon}\right)^n$$

where $C_0, \ldots, C_{n-1}$ are positive constants, which depend only on k, d_i and the maximal frequency λ of the quasipolynomial p.

Proof. The sublevel set A_ρ is defined via the real exponential trigonometric quasipolynomial $q(x) = |p(x)|^2$, i.e. $A = A_\rho(p) = \{x \in Q_1^n : q(x) \leq \rho^2\}$. Let us subdivide Q_1^n into adjacent ε-cubes Q_ε with respect to the standard Cartesian coordinate system. Each Q_ε having a nonempty intersection with A, is either entirely contained in A, or it intersects the boundary ∂A of A. Certainly, the number of those boxes Q_ε, which are entirely contained in A, is bounded by $\mu_n(A)/\mu_n(Q_\varepsilon) = \mu_n(A)/\varepsilon^n$. In the other case, where Q_ε intersects ∂A, it means that there exist faces of Q_ε that have a non-empty intersection with ∂A. Among all these faces, let us take the one with the smallest dimension s. In other words, there exists an s-face F of the smallest dimension s that intersects ∂A, for some $s = 0, 1, \ldots, n$. Let us fix

an s-dimensional affine subspace V, which corresponds F. Then F contains completely some of the connected components of $A \cap V$, otherwise ∂A would intersect a face of Q_ε of a dimension strictly less than s. Clearly, inside each compact connected component of $A \cap V$ there is a critical point of q, which is defined by the system of equations $\frac{\partial q(x)}{\partial x_{j_1}} = \cdots = \frac{\partial q(x)}{\partial x_{j_s}} = 0$ (assuming that V is a parallel translation of the coordinate subspace in $\mathbb{R}^n$ generated by $x_{j_1}, \ldots, x_{j_s}$). After a small perturbation of q we can always assume that all such critical points are non-degenerate. Hence by Lemma 3.4 the number of these points, and therefore of the boxes Q_ε of the considered type, is bounded by $\hat{C}_s \lambda^s$. According to the partitioning construction of Q_1^n, we have at most $\left(\frac{1}{\varepsilon}+1\right)^{n-s}$ s-dimensional affine subspaces with respect to the same s coordinates. On the other hand, the number of different choices of s coordinates is $\binom{n}{s}$. It means the number of boxes that have an s-face F, which contains completely some connected component of $A \cap V$, is at most $\binom{n}{s} \cdot \left(\frac{1}{\varepsilon}+1\right)^{n-s} \hat{C}_s \lambda^s$, which does not exceed, assuming $\varepsilon \leq 1$, the constant $C_{n-s} := \binom{n}{s} 2^{n-s} \hat{C}_s \lambda^s (\frac{1}{\varepsilon})^{n-s}$. Note that C_0 is the bound on the number of boxes that contain completely some of the connected components of A. Thus, we have

$$M(\varepsilon, A) \leq C_0 + C_1 \left(\frac{1}{\varepsilon}\right) + \cdots + C_{n-1} \left(\frac{1}{\varepsilon}\right)^{n-1} + \mu_n(A) \left(\frac{1}{\varepsilon}\right)^n$$

This completes our proof. ■

4 Metric span and generalized Brudnyi's inequality

Let p be a quasipolynomial as above, with the parameters n, k, d_j. These parameters, together with the maximal frequency λ of p form the multi-dimensional diagram D of p. Notice that in contrast to the one-dimensional case (and with Theorem 3.1) we restrict ourselves to the unit box Q_1^n. So B does not appear in the diagram. For a given $0 < \varepsilon \leq 1$ let us denote by $M_D(\varepsilon)$ the quantity $M_D(\varepsilon) = \sum_{j=0}^{n-1} C_j (\frac{1}{\varepsilon})^j$, where $C_0, \ldots, C_{n-1}$ are the constants from Lemma 3.5. Extending the terminology from the one-dimensional case above, we call $M_D(\varepsilon)$ the "frequency bound" for D. Note that the constants C_j depend only on the parameters n, k, d_i and on the maximal frequency λ

of the quasipolynomial p. By Lemma 3.5 for any sublevel set A_ρ of p we have

$$M(\varepsilon, A) \leq M_D(\varepsilon) + \mu_n(A)\left(\frac{1}{\varepsilon}\right)^n$$

Now for any subset $\Omega \subset Q_1^n$ we introduce the metric span ω_D of Ω with respect to a given diagram D as follows:

Definition 4.1. For a subset $\Omega \subset \mathbb{R}^n$ the metric span ω_D is defined as

$$\omega_D(\Omega) = \sup_{\varepsilon>0} \varepsilon^n [M(\varepsilon, \Omega) - M_D(\varepsilon)].$$

Lemma 4.1. *Let $A \subset Q_1^n$ be a sublevel set of a real quasipolynomial with the diagram D. Then for any $\Omega \subset A$ we have*

$$\mu_n(A) \geq \omega_D(\Omega).$$

Proof. This fact follows directly from Lemma 3.5. Indeed, for any $\varepsilon > 0$ we have

$$M(\varepsilon, \Omega) \leq M(\varepsilon, A) \leq M_D(\varepsilon) + \mu_n(A)\left(\frac{1}{\varepsilon}\right)^n.$$

Consequently, for any $\varepsilon > 0$ we have $\mu_n(A) \geq \varepsilon^n [M(\varepsilon, \Omega) - M_D(\varepsilon)]$. Now, we can take the supremum with respect to ε. ∎

For some examples and properties of sets in $\mathbb{R}^n$ with positive metric span, see [10, Section 5]. Here we mention only that for a measurable $\Omega \subset \mathbb{R}^n$ we always have $\omega_D(\Omega) \geq \mu_n(\Omega)$. The proof is exactly the same as in the remark after Theorem 1.2. Now we can prove our generalization of Brudnyi's Theorem 3.1 above.

Theorem 4.2. *Let p be as above and let $\Omega \subset Q_1^n$. Then*

$$\sup_{Q_1^n} |p| \leq \left(\frac{cn\mu_n(B)}{\omega_D(\Omega)}\right)^\ell \cdot \sup_{\Omega} |p|.$$

Proof. Let $\hat{\rho} := \sup_\Omega |p|$. For the sublevel set $A_{\hat{\rho}}$ of the quasipolynomial p we have $\Omega \subset A_{\hat{\rho}}$. By Lemma 4.1 we have $\mu_n(A_{\hat{\rho}}) \geq \omega_D(\Omega)$. Now since p is bounded in absolute value by $\hat{\rho}$ on $A_{\hat{\rho}}$ by definition, we can apply Theorem 3.1 with $B = Q_1^n$ and $A_{\hat{\rho}}$. This completes the proof. ∎

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